Written Exam for the M.Sc. in Economics 2010 (Fall Term)

Financial Econometrics A: Volatility Modelling

Final Exam: Masters course

Exam date: **18/2-2011**

3-hour open book exam.

Notes on Exam: Please note that there are a total of 8 questions which should all be answered. These are divided into Question 1 (Question 1.1-1.4) and Question 2 (Question 2.1-2.4).

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

August Exam Financial Econometrics A, Spring 2009.

Exam Question 1:

From the rich literature on ARCH models an asymmetric ARCH (AARCH) model for log-returns x_t has been proposed and we shall study this in this question.

It is given by,

$$x_t = \sigma_t z_t \qquad \sigma_t = \omega + \alpha|x_{t-1}| + \beta|x_{t-1}|s_t \qquad \text{with } s_t = 1\left(x_{t-1} < 0\right),$$

with z_t iidN$(0,1)$ for $t = 1, ..., T$ and with x_0 fixed. Moreover, the A-ARCH parameters ω, α and β are positive, that is, $\omega > 0$, $\beta > 0$ and $\alpha > 0$.

Note that the model differs from the classical ARCH by specifying σ_t as as a function of $|x_{t-1}|$ rather than σ_t^2 in terms of x_{t-1}^2. Also the additional skew ARCH effect is only present if the past return is negatíve.

Question 1.1:

Figure 1.1 shows a simulated sample with (true values) $\omega_0 = 1, \alpha_0 = 0.4$ and $\beta_0 = 0.6$. Comment on Figure 1.1 (ARCH effects) and what the model may capture as opposed to a classic linear ARCH(1) model.

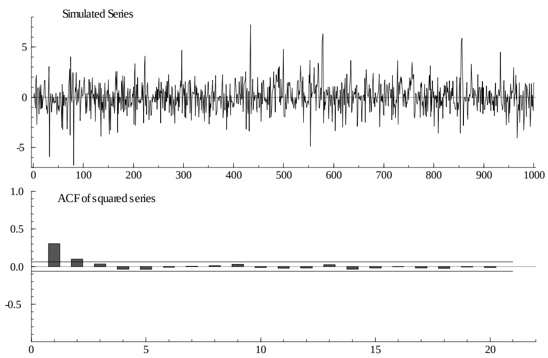

Figure 1.1

Table 1.1 gives output from estimation of the classic ARCH(1) model on the form:

$$\sigma_t^2 = a + bx_{t-1}^2,$$

where a and b are positive parameters. Comment on the output in terms of misspecification.

Table 1.1	
Parameter estimate:	$\hat{b} = 0.96$ (std.deviation $= 0.1$)
Standarized residuals from ARCH(1): $\hat{z}_t = x_t / \hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	6.9 (p-value: 0.03)
LM test for ARCH in $\hat{z}_t$:	0.4 (p-value: 0.65)

Question 1.2: Use the drift function $\delta(x) = 1 + x^2$, and show that for x_{t-1}^2 large,

$$E(\delta(x_t) | x_{t-1}) \leq \phi \delta(x_{t-1}),$$

for some $\phi < 1$ if $\max(\alpha, \beta) < 1$. This implies that x_t is weakly mixing, and hence stationary with $Ex_t^2 < \infty$ if $\max(\alpha, \beta) < 1$.

Question 1.3: Next, turn to estimation of the parameter β (leaving ω and α as fixed or known for simplicity). The log-likelihood function $\ell_T(\beta)$ is (apart from a constant) given by,

$$\ell_T(\beta) = -\frac{1}{2} \sum_{t=1}^{T} \left(2 \log \sigma_t + \frac{x_t^2}{\sigma_t^2} \right).$$

We know from Theorem III.2 (in Part III of the lecture notes), that $\hat{\beta}$ is consistent and asymptotically Gaussian provided regularity conditions hold. A key condition is (A.1) in Theorem III.2 which states that

$$\frac{1}{\sqrt{T}} \partial \ell_T(\beta) / \partial \beta |_{\beta = \beta_0} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(\frac{x_t^2}{\sigma_t^2} - 1 \right) \left(\frac{|x_{t-1}| s_t}{\sigma_t} \right) \xrightarrow{D} N(0, \Omega_S).$$

Show that this holds. Be specific about which results you use and why they apply.

We can then conclude (do not show this) that

$$\sqrt{T} \left(\hat{\beta} - \beta_0 \right) \xrightarrow{D} N\left(0, \Omega_S^{-1}\right).$$

Question 1.4: The A-ARCH model is applied to the FTSE log-return series studied in lectures for $T = 1516$ daily returns. Table 1.2 gives the output from estimation of the A-ARCH model.

Table 1.2	
Parameter estimates:	$\hat{\alpha} = 0.41$ (std.dev. $= 0.04$)
	$\hat{\beta} = 0.15$ (std.deviation $= 0.03$)
Standarized residuals: $\hat{z}_t = x_t / \hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	1563 (p-value: 0.00)
LM test for ARCH in $\hat{z}_t$:	10.5 (p-value: 0.001)

Comment on the size of $\hat{\alpha}$ and $\hat{\beta}$ in terms of the theory derived above, and comment on the misspecification tests.

Exam Question 2:

Question 2.1: Consider the $T = 1000$ observations of x_t in Figure 2.1 below. Comment on the level and volatility patterns as well as the ACF. Would a classic GARCH model with constant mean and GARCH effects be suitable?

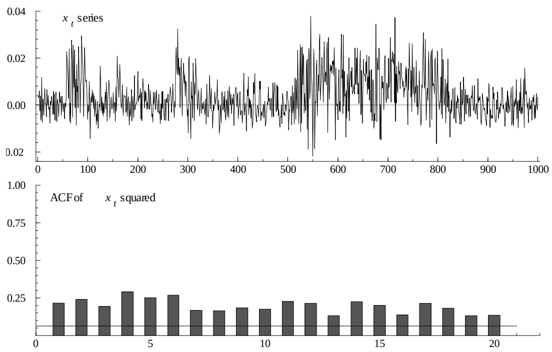

Figure 2.1

Question 2.2: Based on Figure 2.1. consider the following 2-state volatility model for log-returns x_t where both level and volatility changes between two values. Specifically, consider the model given by,

$$x_t = \mu_t + \sigma_t z_t \text{ with}$$
$$\mu_t = \begin{cases} \mu_1 & \text{if } s_t = 1 \\ \mu_2 & \text{if } s_t = 2 \end{cases}, \qquad \sigma_t^2 = \begin{cases} \sigma_1^2 & \text{if } s_t = 1 \\ \sigma_2^2 & \text{if } s_t = 2 \end{cases}$$

and z_t iidN$(0,1)$ distributed and $t = 1, ..., T$. If s_t was observed, it can be shown that (with σ_1^2 known) that the MLE of μ_1, $\hat{\mu}_1$, is given by

$$\hat{\mu}_1 = \frac{\sum_{t=1}^{T} 1\left(s_t = 1\right) x_t}{\sum_{t=1}^{T} 1\left(s_t = 1\right)}.$$

Give an intuitive interpretation of this estimator.

Question 2.3: Next, we assume that s_t is an **unobserved** two-state Markov-chain as given by the transition matrix,

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{pmatrix},$$

with

$$p_{ij} = P\left(s_t = j | s_{t-1} = i\right)$$

such that $p_{11} = 1 - p_{12}$ and $p_{22} = 1 - p_{21}$. With $x_t = \sigma_{s_t} z_t$, z_t iidN$(0,1)$, the complete likelihood function treating s_t as **observed (**and the initial distribution of s_1 as known) is given by $\ell_T\left(\theta\right)$. Set $X_T = \left(x_1, ..., x_T\right)$, and observe that,

$$E\left(\ell_T\left(\theta\right)|X_T\right) = -\frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{2} p_t^*\left(i\right)\left(\log\sigma_i^2 + \frac{\left(x_t - \mu_i\right)^2}{\sigma_i^2}\right)$$
$$-\sum_{t=2}^{T}\sum_{i,j=1}^{2} p_t^*\left(i,j\right)\log p_{ij},$$

where $p_t^*\left(i\right)$ are the *smoothed* probabilities and $p_t^*\left(i,j\right)$ the *smoothed* transitition probabilities.

Maximizing $E\left(\ell_T\left(\theta\right)|X_T\right)$ with respect to μ_1 gives $\tilde{\mu}_1$ when keeping the smoothed probabilities and smoothed transitition probabilities fixed.

Find $\tilde{\mu}_1$ and compare it to $\hat{\mu}_1$ in Question 2.2.

Question 2.4: Discuss under which restrictions on the parameters in the transition matrix P you would expect $\tilde{\mu}_1$ to be a consistent and also asymptotically Gaussian distributed estimator of μ_1.

Estimation of the model with the data in Figure 2.1 gave the output in Table 2.1. Comment.

Table 2.1
Parameter estimates:
$\hat{\mu}_1 = 0.0001$ $\quad \hat{\mu}_2 = 0.1$
$\hat{\sigma}_1 = 0.005$ $\quad \hat{\sigma}_2 = 0.01$
$\hat{\mathbf{P}} = \begin{pmatrix} 0.97 & 0.04 \\ 0.03 & 0.96 \end{pmatrix}$